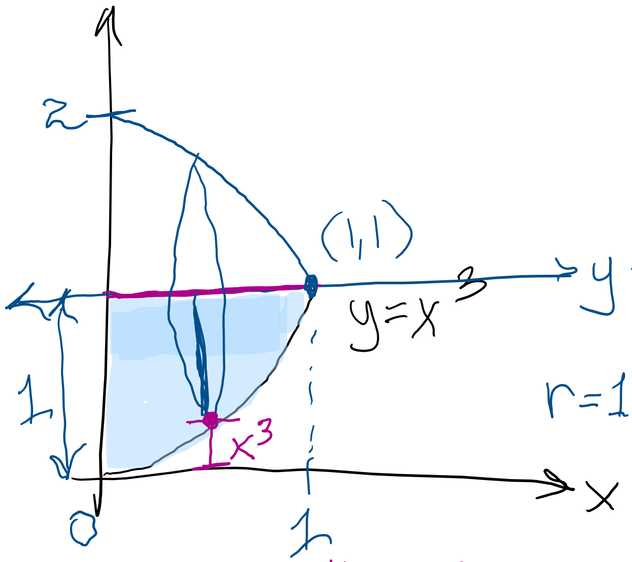
y-axis

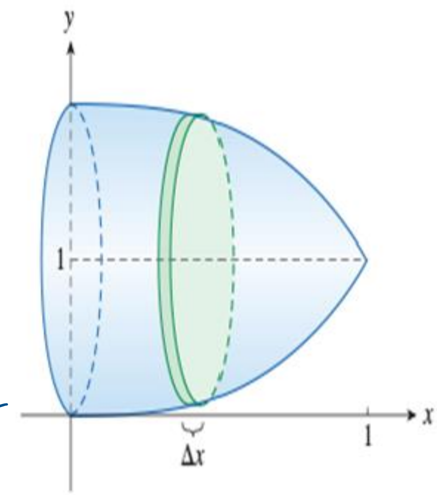
Finish: Find the volume of the solid obtained by rotating the region bound by $y = x^3, y = 1, x = 0$ after being rotated about the line $y = 1$.



disk

$r = 1 - x^3 \Rightarrow A_{SLICE} = \pi r^2$

$\pi(1-x^3)^2$



sum, r^2

$$V = \pi\int_0^1 (1-x^3)^2\,dx$$

$$= \pi\int_0^1 (1-x^3)(1-x^3)\,dx$$

$$= \pi\int_0^1 (1-2x^3+x^6)\,dx$$

$$= \pi\left(x - \frac{x^4}{2} + \frac{x^7}{7}\right)\Big|_0^1$$

$$= \pi\left(1 - \frac{1}{2} + \frac{1}{7} - 0\right)$$

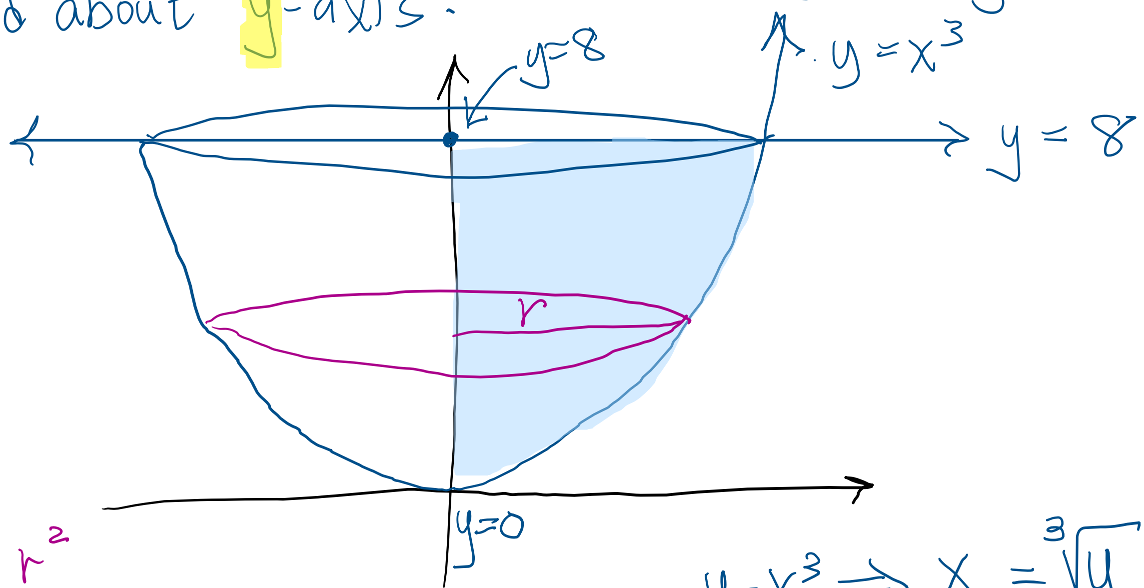
$$= \pi\left(\frac{14}{14} - \frac{7}{14} + \frac{2}{14}\right)$$

$$= \pi \cdot \frac{9}{4} = \boxed{\frac{9\pi}{14}}$$

Again, Sometimes It's Advantageous to Integrate WRT *y*

ex. Find the volume of the solid obtained by rotating the region bound by $y = x^3,\ y = 8,\ x = 0$

rotated about y-axis.



$y = x^3 \Rightarrow x = \sqrt[3]{y}$

$\therefore\ r = \sqrt[3]{y} = y^{1/3}$

$r^2 = (y^{1/3})^2 = y^{2/3}$

$$V = \pi \int_0^8 \overset{r^2}{y^{2/3}}\, dy$$

$$= \frac{3\pi}{5} y^{5/3} \Big|_0^8$$

$$= \frac{3\pi}{5}\left((8^{1/3})^5 - 0 \right)$$

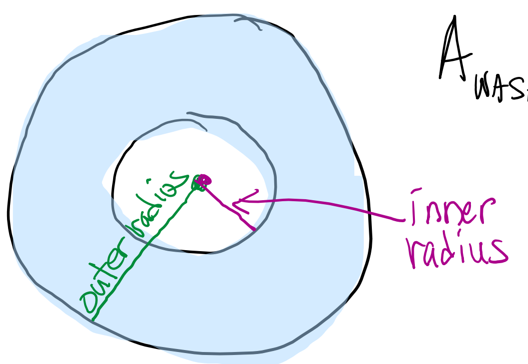
$$= \frac{3\pi}{5}(2^5)$$

$$= \frac{3\pi}{5} \cdot 32 = \boxed{\frac{96\pi}{5}}$$

$8^{1/3} = \sqrt[3]{8} = 2$

$$\begin{array}{r} 32 \\ \times 3 \\ \hline 96 \end{array}$$

Find Volume Using The Washer Method



$$A_{\text{WASHER}} = \underbrace{A_{\text{OUTER CIRCLE}}}_{\pi (r_{\text{OUTER}})^2} - \underbrace{A_{\text{INNER CIRCLE}}}_{\pi (r_{\text{INNER}})^2}$$

$$V_{\text{shape}} = \int_a^b \left[\pi (r_{\text{outer}})^2 - \pi (r_{\text{inner}})^2\right] dx$$

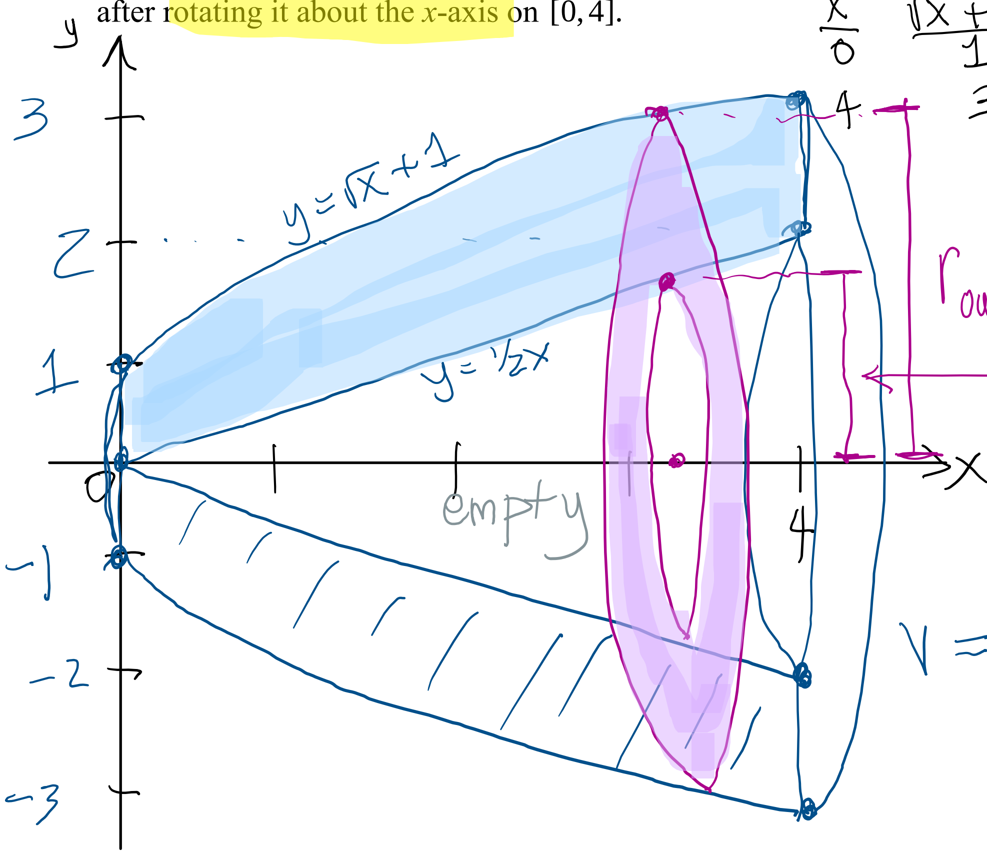
$$\boxed{V = \pi \int_a^b \left[(r_O)^2 - (r_I)^2\right] dx}$$

USING WASHER METHOD

ex. Explore the solid obtained by rotating the region enclosed by $y = \sqrt{x} + 1$, $y = \frac{1}{2}x$

after rotating it about the x-axis on $[0, 4]$.

x	$\sqrt{x}+1$	$\frac{1}{2}x$
0	1	0
4	3	2



$$r_{\text{OUTER}} = \sqrt{x} + 1$$

$$r_{\text{INNER}} = \frac{1}{2}x$$

$$(r_O)^2 - (r_I)^2$$

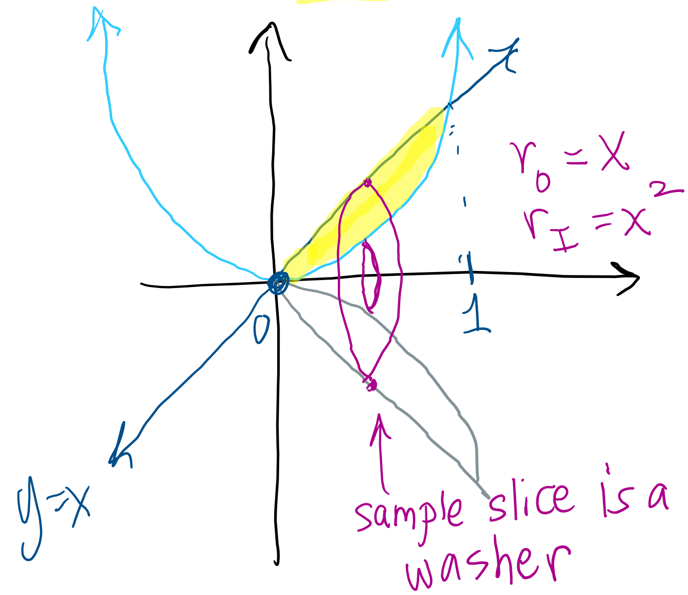
$$V = \pi \int_0^4 \left[(\sqrt{x}+1)^2 - \left(\frac{1}{2}x\right)^2\right] dx$$

https://www.youtube.com/watch?v=R-t6BUMkqRQ ex. Find region enclosed by the curves $y=x$, $y=x^2$ and is rotated about the x-axis then find its volume.

INTERSECTION POINTS:

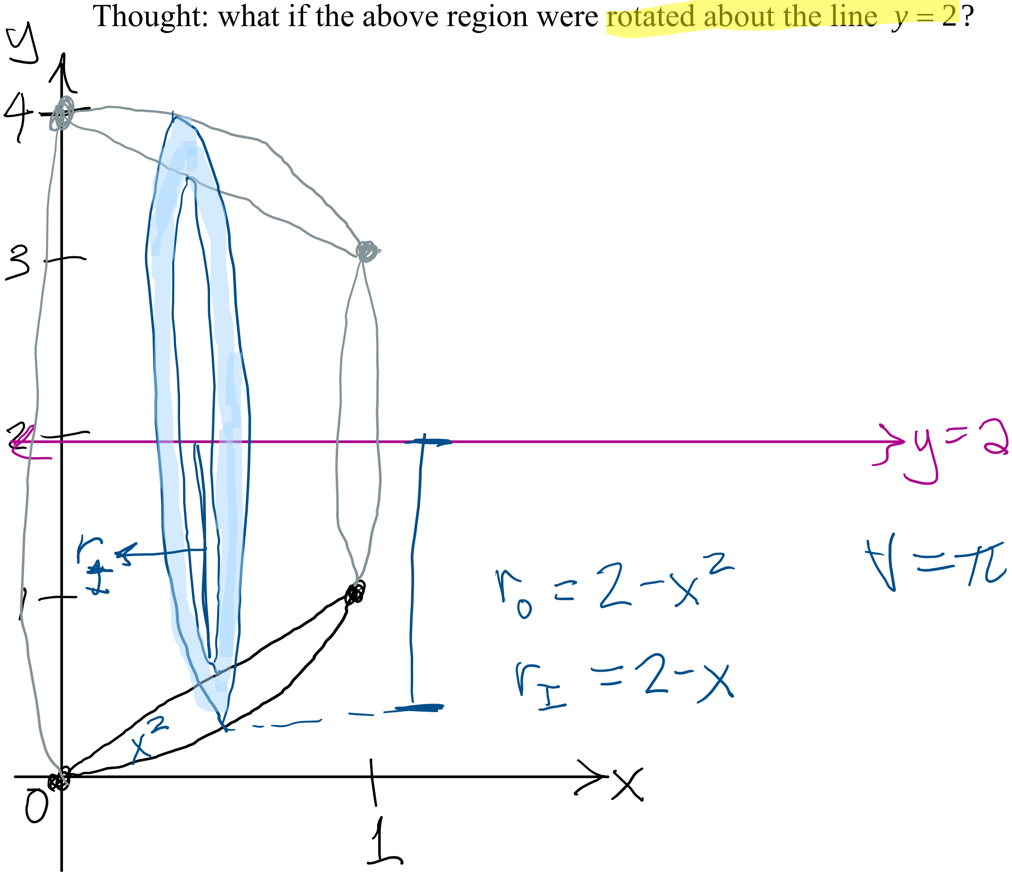
$$x = x^2$$
$$0 = x^2 - x$$
$$0 = x(x-1) \Rightarrow x=0 \quad x=1$$

$$V = \pi \int_0^1 \Big(\underbrace{x}_{r_0}{}^2 - \big(\underbrace{x^2}_{r_I}\big)^2\Big)\,dx$$
$$= \pi \int_0^1 (x^2 - x^4)\,dx \quad \text{Do: finish}$$
$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1$$
$$= \pi\left(\frac{1}{3}\cdot\frac{5}{5} - \frac{1}{5}\cdot\frac{3}{3} - 0\right)$$
$$= \pi\left(\frac{2}{15}\right) = \boxed{\frac{2\pi}{15}}$$



Thought: what if the above region were rotated about the line $y = 2$?

$y=x \quad y=x^2$



$r_0 = 2 - x^2$

$r_I = 2 - x$

$$V = \pi \int_0^1 \Big[\underbrace{(2-x^2)^2}_{r_0^2} - \underbrace{(2-x)^2}_{r_I^2}\Big]\,dx$$

$y = \sqrt{x}$

ex. Find region enclosed by the curves $y^2 = x$, $x = 2y$ and is rotated about the *y-axis* then find its volume.

Intersection Points:

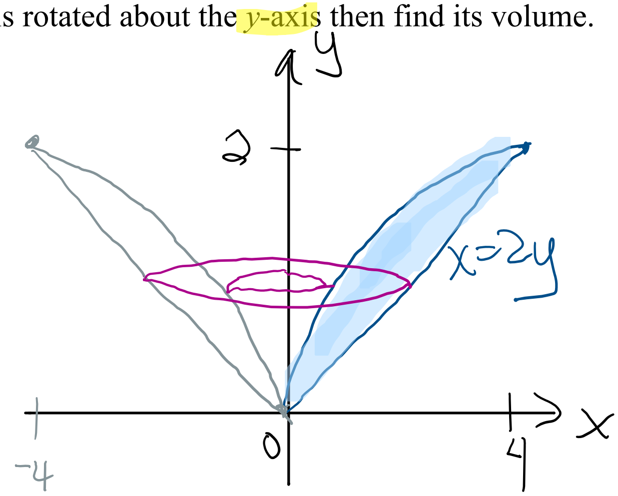
$y^2 = 2y$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0 \quad y = 2$

y	y^2	$2y$
0	0	0
2	4	4



Do: find $r_o = 2y \quad r_o^2 = 4y^2$

$r_I = y^2 \quad r_I^2 = y^4$

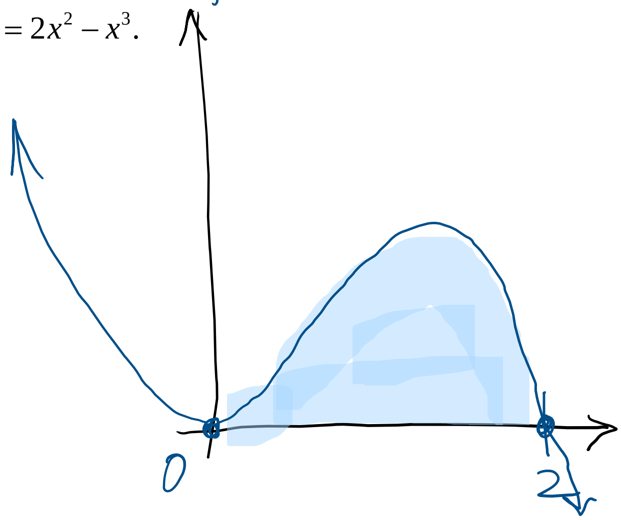
$$V = \pi \int_0^2 (\overset{r_o^2}{4y^2} - \overset{r_I^2}{y^4})\,dy = \boxed{\frac{64\pi}{15}}$$

$-x^3$

Find Volume Using Cylindrical Shells

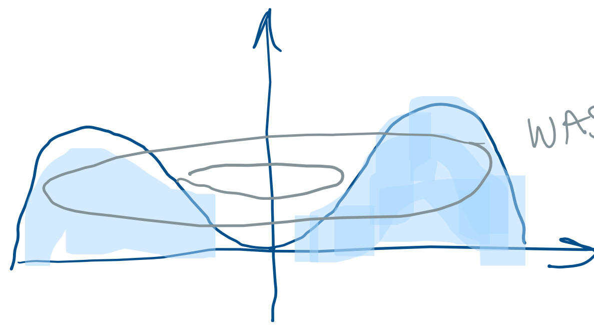
Sometimes volumes are impractical (if not impossible) to find using disk or washer method.

ex. Sketch region bound by $x = 0$, $y = 0$ and $y = 2x^2 - x^3$.

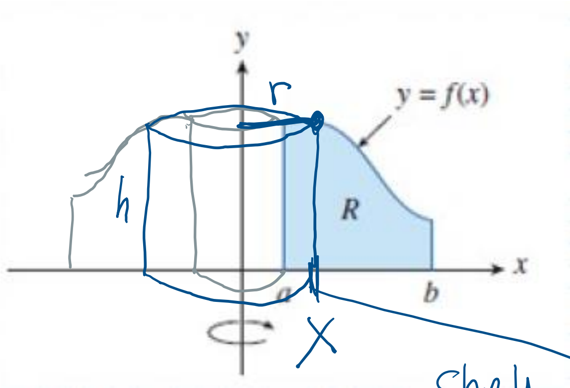


Next, rotate region about y-axis.

alternate method: use cylindrical shells as slices



WASHER!



$V_{CYL} = \pi r^2 h$

$r = x$

$h = f(x)$

shell

$h = f(x)$

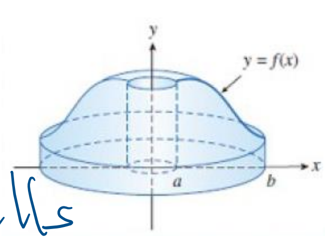
$C = 2\pi x$ ← r

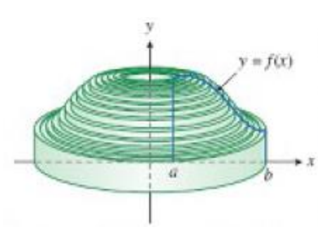
Δx

$$V = \int_a^b 2\pi x f(x)\,dx$$

$$\boxed{V = 2\pi \int_a^b x f(x)\,dx}$$

volume using cylindrical shells

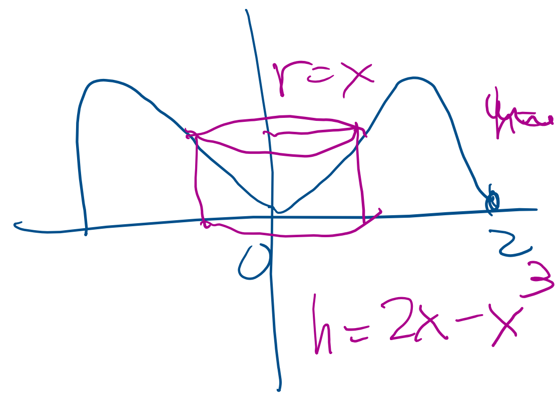




ex. Find the volume of solid obtained by region bound by $x=0$ $y=0$ $y=2x-x^3$

rotate about y-axis using cylindrical shells.

$$V = 2\pi \int_0^2 \underbrace{x}_{r}\underbrace{(2x-x^3)}_{h}\,dx$$



$$= 2\pi \int_0^2 (2x^2 - x^4)\,dx$$

$$= 2\pi \left(\frac{2x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2$$

$$= \boxed{\frac{16\pi}{15}}$$